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Mathematical objects and sense
How semiotic transformations change the sense of mathematical objects [1]

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Sunto. In questo articolo intendo mostrare una conseguenza che si manifesta talvolta nelle trasformazioni semiotiche di trattamento e conversione di una rappresentazione semiotica il cui senso deriva da una pratica condivisa; il passaggio dalla rappresentazione di un oggetto matematico ad un'altra attraverso trasformazioni, da un lato conserva il significato dell'oggetto stesso, ma talvolta può cambiarne il senso. Questo fatto viene dettagliatamente mostrato attraverso un esempio, ma

[1] A shorter version of this article can be found in:
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inserendolo all'interno di una vasta cornice teorica che chiama in causa gli oggetti matematici, i loro significati, le loro rappresentazioni.

Summary. In this paper I will demonstrate a consequence at times manifest in the semiotic transformations involving the treatment and conversion of a semiotic representation whose sense derives from a shared practice. The shift from one representation of a mathematical object to another via transformations, on the one hand maintains the meaning of the object itself, but on the other can change its sense. This is demonstrated in detail through a specific example, while at the same time it is collocated within a broad theoretical framework that poses fundamental questions concerning mathematical objects, their meanings and their representations.

Resumé. Dans cet article j'entends montrer une conséquence qui se manifeste quelquefois dans les transformations sémiotiques de traitement et de conversion d'une représentation sémiotique, dont le sens découle d'une pratique partagée; le passage d'une représentation d'un object mathématique à une autre, au moyen de transformations, d'un côté il conserve la signification de l'object pris en considération, mais il peut aussi changer son sens. Ce fait peut être montré par un exemple, mis à l'intérieur d'un cadre théorique, qui prend en considération les objects mathématiques, leurs significations, leurs représentations.

Zusammenfassung. In diesem Artikel möchte ich eine Folgerung, die sich manchmal in den semiotischen Veränderungen der Behandlung und der Umwandlung einer semiotischen Dartstellung erweist, deren Sinn von einer billigten Praxis stammt; der Übergang mittels Veränderungen von der Darstellung eines matematischen Gegenstandes zu einer anderen Darstellung bewahrt einerseits den Sinn des Gegenstandes selbst, aber kann es manchmal die Bedeutung ändern. Dies wird ausführlich durch ein Beispiel gezeigt, welches aber in einem grossen theoretischen Rahmen eingereiht wird, der sich die mathematischen Gegenstände, sowie deren Bedeutungen und Darstellungen bedient.

Resumen. En este artículo intento mostrar una consecuencia que se evidencia algunas veces en las transformaciones semioticas de tratamiento y conversión de una representación semiotica cuyo sentido deriva de una práctica compartida; el pasaje de la representación de un objeto matemático a otra por medio de transformaciones, de una parte conserva el significado del objeto mismo, pero en ocasiones puede cambiar su sentido. Esto hecho está aquí evidenciado detalladamente por medio de un ejemplo, pero insertándolo en el seno de un amplio marco teórico que llama en causa los objetos matemáticos, sus significados, sus representaciones.

Resumo. Neste artigo quero mostrar uma consequência que, ás vezes, se apresenta nas trasformações semióticas de processamento e conversão de uma representação semiótica, o sentido da qual resulta de uma pratica compartilhada; a passagem da representação de um objeto matemático para uma outra através de trasformações, mantêm o significado do mesmo objeto, mas ás vezes pode trocar o seu sentido. Isso è mostrado com detalhes através de um exemplo, inscrito dentro de um grande quadro teórico que o relaciona com os objetos matemáticos, seus significados, suas representações.

0. Premise

This study is divided in two distinct parts.

In the first part, principally via pertinent quotations, I will describe a general epistemological, ontological and semiotic path within the currently much-debated question of certain theoretical approaches to research in Mathematics education. The aim is to clearly delimit and circumscribe the theoretical field of reference and avoid any possible theoretical misunderstanding.

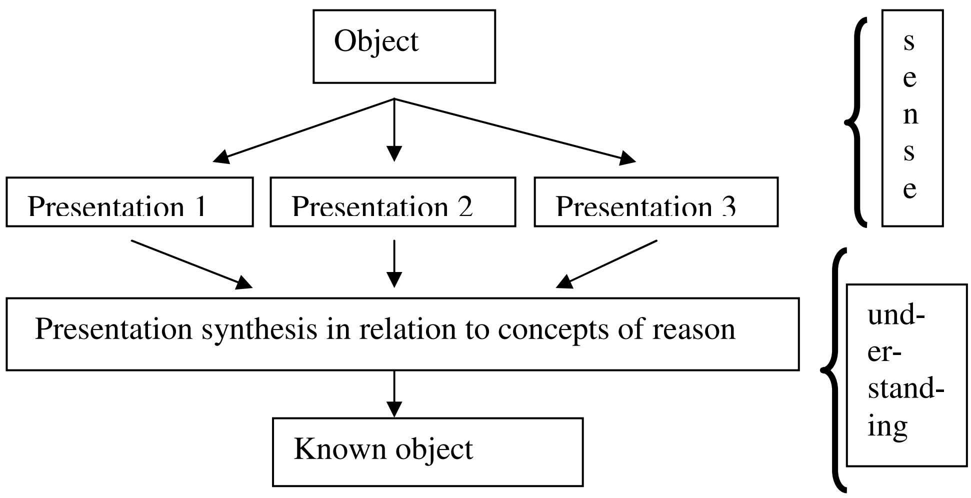
In the second part, via the narration of one principal and several other exemplary episodes, I will discuss a question, subsequently central to my conclusion, concerning the attribution of different senses to diverse semiotic representations which could represent the same mathematical object, an attribution on the part of both students and teachers (trainee and in service teachers) at all school levels.

Part 1

1. A theoretical path

1.1. *Ontology and knowledge*
In a number of studies in the late 1980s and 1990s I sustained the position that, while the mathematician can avoid debating the question of the *sense* of the *mathematical objects* he uses and of the sense of *mathematical knowledge*, this question is of vital importance for the researcher in Mathematics education (D'Amore 1999, pp 23-28, and elsewhere). Such a position that I appreciate truly is amply supported by Radford (2004): «One can very well survive doing mathematics without adopting an explicit ontology, that is, a theory dealing with the nature of mathematical objects. (...) The situation has become very different when we talk about *mathematical knowledge*. (…) Theoretical questions about the content of knowledge and the ways such a content is transmitted, acquired or constructed, has led us to a point in which we can no longer avoid taking ontology seriously».
This conviction has led me to dedicate much time to the study of conceptual knowledge, after having established an ontological belief on the basis of the *way* in which human beings *know* concepts (D'Amore, 2001a,b; 2003a,b). The debate is long-standing and can be traced back to Ancient Greece, but Radford makes every effort to pose the question in modern terms: «Men, he said, have a prior intellectual knowledge of conceptual things thanks to an autonomous activity of

the mind, independently of the concrete world» (Radford, 2004) [the reference "he said" is to the mathematician Pietro Catena (1501-1576, Professor at the University of Padua and author of *Universa Loca*, in which he asserted that «mathematical objects are ideal and innate entities» (Catena, 1992)].
The debate becomes truly modern with the distinction between (human) "intellectual concepts" and "concepts of objects" proposed by Immanuel Kant (1724-1804) in his *Critique of Pure Reason*: «[These] concepts of the pure intellect are not concepts of objects; they are logical schemata without content; their function is to make possible a regrouping or *synthesis* of intuitions. The synthesis is the responsibility of what Kant identified as the cognitive faculty of Understanding» (Radford, 2004).
The following diagram is a particularly interesting attempt to illustrate the correct relationship between the idea of *sense* and of *understanding*:



1.2. *An anthropological approach*
Having to take a definite position, for many researches including myself, an almost obliged choice has gone through the following line of development: from an anthropological approach to a pragmatic choice (D'Amore, 2003b, and elsewhere). Once again, the position of Radford is clearly within this tradition: «In this line of thought, we cannot avoid taking into account an anthropological approach (…) the fact that the manners in which we use the diverse kinds of signs and artefacts during our acts of knowing are subsumed in cultural

prototypes of mediated meanings (…). What is relevant is that the use of signs and artefacts alter our modes of reception of the objects of the world, that is to say, signs and artefacts alter the way in which the objects are given to us through our senses (…). To summarize: From the viewpoint of an anthropological epistemology, the way in which I see that the riddle of mathematical objects can be solved is to consider mathematical objects as fixed patterns of activity embedded in the always changing realm of reflective and mediated social practice» (Radford, 2004).

There is general convergence of opinion concerning this position: «Mathematical objects must be considered symbols of cultural units which emerge through a system of uses connected to mathematical activities practiced by groups of people and thus evolve with the passage of time. In our conception, what determines the progressive emergence of "mathematical objects" is the fact that certain types of practices are typical of specific institutions and that the "meaning" of these objects is intimately linked to the problems faced and the activities conducted by human beings, since it is –impossible the reduction of the meaning of a mathematical object merely to its mathematical definition» (D'Amore, Godino, 2006).

1.3. *Systems of practice*

This convergence can be further exemplified: «The notions of "institutional (and) personal meaning" of mathematical objects have led to those of "personal practice", "systems of personal practices", "personal (or mental) object", useful instruments for the study of "individual mathematical cognition" (Godino, Batanero, 1994; 1998). Each of these notions has a precise institutional collocation. Clarifying these points is essential in order to define and render operative the notions of "personal and institutional relationship to the object" introduced by Chevallard (1992)» (D'Amore, Godino, 2006).

Our idea of "system of personal practices" is consistent with Radford's anthropological semiotic approach (ASA): «In the anthropological semiotic approach (ASA) the ideality of the concept of the conceptual objects is directly connected to the historical and cultural context. The ideality of mathematical objects – i.e. what makes them general – is entirely dependent on human activity» (Radford, 2005).

The sociological aspects of this dependence on human activity and social practice is thus expressed: «The mathematical learning of an object O by an individual I within the society S is nothing more than the agreement of I to the practices that other members of S develop with reference to the object O» (D'Amore, in D'Amore, Radford, Bagni, 2006) and: «classroom practices can be considered as systems of adaptation of students to society» (Radford, in D'Amore, Radford, Bagni, 2006).

1.4. *Object and mathematical object*

Nevertheless, this "mathematical object" has to be defined: the definition we propose is a generalization of Blumer's (1969, pag. 8) idea: «*Mathematical object* (Godino, 2002): anything that can be indicated, pointed out, named during mathematical construction, communication or learning; an object is "anything that can be indicated, pointed out or to which one can refer".

We can distinguish different types of mathematical objects at various levels:

- "language" (terms, expressions, notations, graphs, …) in various registers (written, oral, gestural, ….)
- "situations" (problems, extramathematical applications, exercises, …)
- "actions" (operations, algorithms, techniques for calculating, procedures, …)
- "concepts" (introduced via definitions or descriptions) (line, point, number, mean, function, …)
- "properties or attributes of objects" (propositions concerning concepts, …)
- "argumentations" (for example, the validation or explanation of propositions, deductions, etc. …).

These objects are then organised within more complex entities such as conceptual systems, theories, …» (D'Amore, Godino, 2006).

In the quoted work we exploit the notion of «*semiotic function*, in which a relationship is established between two (ostensible or non ostensible) mathematical objects based upon a representational or instrumental dependence, whereby one can be used in place of the other or one is placed instead of the other (D'Amore, Godino, 2006).

Furthermore, «(…) the mathematical objects referred to in mathematical practices and their emergents, on the basis of the linguistic game of which they are a part, can be considered in terms of the following dual aspects or dimensions (Godino, 2002):

- personal – institutional: as we have already seen, shared systems of practices within an institution give rise to emergent objects that are considered "institutional objects", while within systems used by a single individual can be considered as "personal objects";

- ostensible (graphs, symbols, …) – non ostensible (which evoke “doing” Mathematics, represented in texts, oral, graphic, gestural, …);
- extensive – intensive: the relationship established between an object introduced in a linguistic game as a specific, *concrete* example (for example, the function $y=2x+1$) and a more general, *abstract* class (for example, the family of functions $y= mx+n$);
- elementary – systemic: in some circumstances mathematical objects function as unitary entities (presumably already known) while in others they function as systems which can be broken down for analysis;
- expression – content: prior and subsequent to any semiotic function.

These aspects are presented in complementary pairs which exist in a dual and dialectic manner and are considered as attributes applicable to distinct primary and secondary objects, thereby giving rise to distinct “versions” of such objects» (D’Amore, Godino, 2006).
If, however, we consider the linguistic practice of representation: «I think that we must distinguish between two types of objects within the creation of mathematical competence (mathematical learning): the mathematical object itself and the linguistic object that expresses it» (D’Amore, in D’Amore, Radford, Bagni, 2006).
I shall turn back to the representation soon, in order to investigate its roles more specifically.

1.5. *Learning objects*
During my attempts to summarize learning difficulties concerning the concepts and the knowledge of objects, I have often made use of *Duval’s paradox*: «(…) on the one hand the learning of mathematical objects cannot but be a conceptual learning, while on the other activity involving mathematical objects is only possible via the use of semiotic representations. This paradox can constitute a definite vicious circle for the learning process. How can learners avoid confusing mathematical objects with their semiotic representations if they cannot but have a relationship only with semiotic representations? The impossibility of direct access to mathematical objects without all semiotic representations makes their confusion practically inevitable. Moreover, how can learners fully acquire mathematical treatments, necessarily linked to semiotic representations, without a previous conceptual learning of the objects represented? The paradox becomes even stronger if mathematical and conceptual activity are considered

as one and the semiotic representations are then considered secondary or extrinsic» (Duval, 1993, p. 38).
These questions can be mainly referred to a certain way of conceiving the idea of semiotics.
Once again, I agree with Radford: «The epistemological problem can be summarised in the following question: how can we know these general objects when our only access to them is through the representations that we make of them?» (Radford, 2005).

1.6. *The representation of objects*
As regards the representation of objects, I exploit Radford's good summary of Kant's thought: «In a famous letter to Herz, written in February 21, 1772, Kant questions the efficacy of our representations and asks: "On what basis do we construct the relationship between what we call representation and its corresponding object?" (…) In this letter Kant questions the legitimacy of our representations in presenting and representing objects. In semiotic terms, Kant reflects on the adequacy of the sign. (…) Kant's doubt is of an epistemological order» (Radford, 2005).
The question posed particularly concerns the idea of the sign, since for Mathematics this form of representation is specific. The sign is in itself a specification of the particular, but can also be interpreted giving sense to the general: «If a mathematician has the right to perceive the general in the particular, this is, as Duval (1951, p.10) observes, "because he is certain of the faithfulness of the sign as the adequate representation of the meaning"» (Radford, 2005).
Signs are, however, artefacts, linguistic objects (in the broad sense), terms which represent in order to indicate: «(…) objectification indicates a process the scope of which is to show something (an object) to someone. What are the means of showing the object? They are those which I call *semiotic means of objectification*. They are objects, artefacts, linguistic terms, more generally signs used to render visible an intention and conclude an action» (Radford, 2005).
These means perform a multiple role, that I leave out, concerning highly complex interrelationships between sign, culture and humanity: «(…) the entire culture can be seen as a system of systems of signs in which the signified of a signifier becomes in turn a signifier of another signified or indeed the signifier of its own signified» (Eco, 1973, p. 156).
Moreover, the "cognitive role of the sign" is very important (Wertsch, 1991; Kozoulin, 1990; Zinchenko, 1985): I cannot examine closely this aspect in the present paper, although I consider it as a

fundamental concept of General Semiotics: «*all processes of signification between human beings (…) presuppose a system of signification as a necessary condition*» (Eco, 1975, p. 20), i.e. a cultural agreement to codify and interpret and thus produce knowledge.

The choice of signs, above all when composing languages, is neither neutral or independent , but rather preconstitutes the destiny of the thought expressed and of the communication realised. For example, «The language of algebra imposes a sobriety of thought and expression, a sobriety in ways of creating meaning unthinkable before the Renaissance. It imposes what I have elsewhere called a *semiotic contraction* and presupposes the loss of the *origo*» (Radford, 2005).

The loss of *origo* (origin, principle) has been widely studied by Radford (2000, 2002, 2003) and *this loss* constitutes the point of departure for the second part of this paper.

Part 2

2. Object, its shared meaning and its semiotic representations: the narration of an episode

2.1. *The episode*

In a fifth class (pupils aging 10 years) of an Italian Primary School, the teacher has conducted an introductory lesson in a-didactic situation concerning the first elements of probability, in which the pupils construct, with at least the use of some examples, the idea of "event" and "the probability of simple events". As an example, the teacher uses a normal die with six faces to study the random results from a statistical point of view. From this emerges a frequency probability which is, however, interpreted in the classical sense. The teacher then proposes the following exercise:

Calculate the probability of the following event: the outcome of an even number when throwing the die.

Pupils discuss in groups and above all sharing practices devised under the direction of the teacher decide that the answer is expressed by the fraction $\frac{3}{6}$ because «the possible results are 6 (at the denominator) while the results that render the event true are 3 (at the numerator)».

After having institutionalised the construction of this knowledge, satisfied by the result of the experience and the fact that the outcome has been rapidly obtained and the pupils have shown considerable skill in handling fractions, the teacher proposes that, on the basis of the equivalence between $\frac{3}{6}$ and $\frac{50}{100}$, it is also possible to express the probability by writing 50% and that this is indeed more expressive, since it means that the probability of such a result is a half, in terms of the generality of all possible events which is 100. A pupil observes that «so we can also use the [fraction] $\frac{1}{2}$», and the proposal is validated through the statements of the pupil, rapidly accepted by all and once again institutionalized by the teacher.

2.2. *Semiotic analysis*

If we analyse the different semiotic representations of the same event which emerge during this activity – "the result of throwing a die is an even number" – it is possible to identify at least the following:

- semiotic register: natural language: probability that the result of throwing a die is an even number
- semiotic register: the language of fractions: $\frac{3}{6}$, $\frac{50}{100}$, $\frac{1}{2}$
- semiotic register: the language of percentages: 50%.

2.3. *The sense shared via different semiotic representations*

Each of the preceding semiotic representations is the signifier which is *downhill* of the same signified which is *at the top* (Duval, 2003). The shared "sense" of what was being developed together was always the same and therefore the mathematical practice carried out and described led to semiotic transformations for which the final results were easily accepted:

- conversion: from the semiotic representation expressed in the natural language register to the writing $\frac{3}{6}$
- treatment: from the writings $\frac{3}{6}$ and $\frac{50}{100}$ to $\frac{1}{2}$
- conversion: from the writing $\frac{50}{100}$ to 50%.

2.4. *Required previous knowledge*

In the episodes considered several types of knowledge, apparently well-constructed, interact:

- knowledge and use of fractions

- knowledge and use of percentages
- knowledge and use of the event: the outcome of throwing a die is an even number.

Each of these is manifest in the unitary and shared practices of the class.

2.5. *Sequel to the episode: the loss of a shared sense caused by semiotic transformations*

At the end of the sequence the pupils are asked if the fraction $\frac{4}{8}$ can be used to represent the same event, since it is equivalent to $\frac{3}{6}$. *The answer is negative, unanimous and without hesitation*. Even the teacher, who had previously handled the situation with confidence, asserts that «$\frac{4}{8}$ *cannot* represent the event because a die has 6 faces and not 8». Pressed to consider further the question, the teacher adds: «There are not only dice with 6 faces, but also dice with 8 faces. In that case, yes, the fraction $\frac{4}{8}$ represents the outcome of an even number when throwing a die».

Now I am going to examine this episode from a semiotic perspective, after having first generalized the issue.

3. A symbolism for semiotic principles

In other studies I have already used the following definitions and symbols (D'Amore, 2001a, 2003a,b, and elsewhere):

semiotics $=_{df}$ representation realised via a system of signs
noetics $=_{df}$ conceptual acquisition of an object.[2]

Hereafter I will use:

r^m $=_{df}$ m^{th} semiotic register
$R^m_i(A)$ $=_{df}$ i^{th} semiotic representation of concept A in the semiotic register r^m
(m = 1, 2, 3, …; i = 1, 2, 3, …).

[2] According to Plato noetics refers to the act of conceptualising via thought, while according to Aristotle it is the very act of conceptual understanding.

From this we can see that, if the semiotic register changes then so does the semiotic representation, but changing the semiotic representation does not necessarily imply changing the semiotic register.
The following diagram illustrates the question even more clearly (with reference to Duval, 1993):

characteristics of semiotics: *representation – treatment – conversion* [imply different cognitive activities]

concept A to be represented ⇒ choice of distinctive features of A

⇓

REPRESENTATION of A [$R^m_i(A)$] in a given semiotic register r^m

⇓ transformation of representation *TREATMENT*

new representation ($i \neq j$) [$R^m_j(A)$] in the *same* semiotic register r^m

⇓ transformation of register *CONVERSION*

new representation ($h \neq i$, $h \neq j$) [$R^n_h(A)$] in a *different* semiotic register r^n ($n \neq m$)

($m, n, i, j, h = 1, 2, 3, \ldots$)

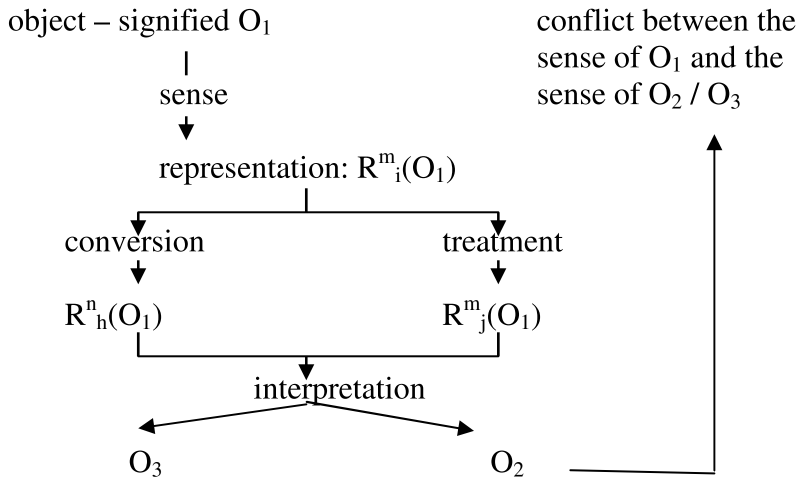
4. Let's turn back to the episode

- There exists a mathematical object (meaning) O_1 to represent: the probability that the result of throwing a die is an even number;
- a *sense* is ascribed to the object on the basis of a presumable shared experience in a social practice which is constructed because shared in the class;
- a semiotic register r^m is chosen in order to represent O_1: $R^m_i(O_1)$;
- a treatment is effected: $R^m_i(O_1) \rightarrow R^m_j(O_1)$;
- a conversion is effected: $R^m_i(O_1) \rightarrow R^n_h(O_1)$;
- $R^m_j(O_1)$ is interpreted and the mathematical object (signified) O_2 is recognised in it;
- $R^n_h(O_1)$ is interpreted and the mathematical object (signified) O_3 is recognised in it.

What is the relationship between O_2, O_3 and O_1?

Identity can be recognised; and this means that there is a previous knowledge, on the basis of which identity itself can be pointed out.
But we can avoid to recognise identity, so the "interpretation" is or seems different, and in this case we lose the *sense* of the original starting-object (signified) O_1.

The following diagram summarises the complexity of what has happened in the classroom, in order to highlight the connection between objects, meanings, semiotic representations and sense:



In our example:

- object - signified O_1: the probability that the result of throwing a die is an even number;
- sense: the shared classroom experience as a classroom practice in a a-didactic situation under the supervision of the teacher leads to the conclusion that the sense of O_1 is that described by the pupils and desired by the teacher: many possible outcomes and many outcomes consistent with the event;
- choice of a semiotic register r^m: rational numbers Q expressed as fractions; representation: $R^m{}_i(O_1)$: $\frac{3}{6}$;
- treatment: $R^m{}_i(O_1) \rightarrow R^m{}_j(O_1)$, i.e. from $\frac{3}{6}$ to $\frac{1}{2}$;
- treatment: $R^m{}_i(O_1) \rightarrow R^m{}_k(O_1)$, i.e. from $\frac{3}{6}$ to $\frac{4}{8}$;
- conversion: $R^m{}_i(O_1) \rightarrow R^n{}_h(O_1)$, i.e. from $\frac{3}{6}$ to 50%;

- $R^m_j(O_1)$ is interpreted and the mathematical object (signified) O_2 is recognised in it;
- $R^m_k(O_1)$ is interpreted and the mathematical object (signified) O_3 is recognised in it;
- $R^n_h(O_1)$ is interpreted and the mathematical object (signified) O_4 is recognised in it.

What is the relationship between O_2, O_3, O_4 and O_1?
In some cases, (O_2, O_4), identity of the signifier is recognised, thus indicating previously-constructed knowledge which permits this recognition. There is one single, shared sense. In another case, (O_3), the identity is not recognised, in that the "interpretation" is or seems to be different and so the sense of the object (signified) O_1 has been lost.
Duval too treats the question of different representation of the same object (Duval 2005).

It is not necessarily the case that the loss of sense occurs only as a result of conversion. As we have seen in our example, the loss is caused by the treatment from $\frac{3}{6}$ to $\frac{4}{8}$. The teacher's interpretation of $\frac{4}{8}$ did not consider a plausible object the very same O_1 derived from the shared sense which had led to the representation $\frac{3}{6}$.

The same experiment conducted with older students and even trainee teachers of primary and secondary school shows that if the treatment from $\frac{3}{6}$ to $\frac{4}{8}$ is an example of loss of sense, the loss is even greater with the treatment from $\frac{3}{6}$ to $\frac{7}{14}$; while it is decidedly less in the conversion from $\frac{3}{6}$ to 0.5.

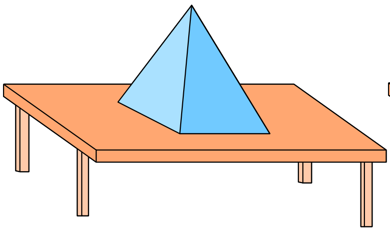
5. Examples of other episodes

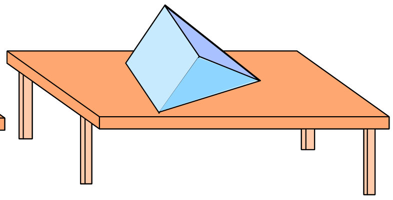
5.1. Nursery school pupils

5

5

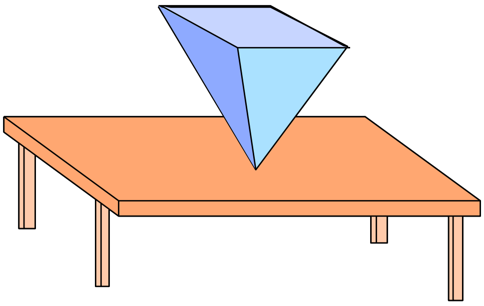
«This is a big number» «This is a small number»





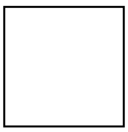
«This is a pyramid» «This is a pyramid lying down»

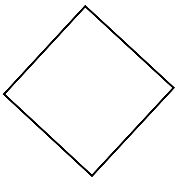
«This isn't a pyramid. It can't stand up!»



5.2. Primary school pupils

«7+3=10 is an addition, but 10=7+3 isn't»





«This is a square» «This isn't, it's a rhombus»

«0,5 means a half»; «1:2 is a half too»; «2:4 is 0,5 but it isn't a half»

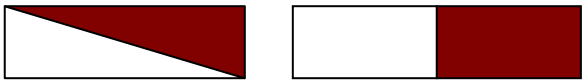
5.3. Middle school pupils

«$\frac{1}{2}$ can be expressed as 0,5 or 50%;

but, while $\frac{1}{2}$ equals to , 0,5 doesn't, and 50% even less»

«$\frac{1}{2}$ is a fraction you use at school, ½ is what you find in books».

«These are two *different* halves of the *same* rectangle»



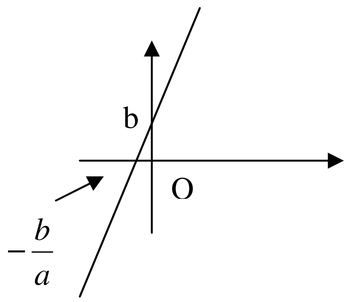
5.4. Secondary school pupils

«A point is a geometric entity which has *zero dimension*; it's a small and round spot; if you change its form it isn't a point any more».

«$y=x^2-2x+1$ is a parabola; (after explicit treatments, $x^2-2x+y+1=0$ is obtained); $x^2-2x+y+1=0$ is *almost* a circumference».

«(…) (x-1)(x+2)=0 isn't an equation, (while) $x^2+x-2=0$ is».

Total cost of *y* € for the rent of a party location for *x* hours at *a* € per hour, plus the fixed cost of *b* €; the students and the teacher produce the semiotic representation: *y=ax+b*; a transformation is effected via the treatment to $x-\frac{y}{a}+\frac{b}{a}=0$ which is represented as:



and universally interpreted as a "straight line". The semiotic representation obtained from the initial representation via treatment and conversion is no longer recognised as the same mathematical object and assumes a different *sense*.

5.5. University students

$$x^2+y^2+2xy-1=0 \xrightarrow{\text{TREATMENT}} x+y=\frac{1}{x+y}$$

sense: from «A circumference» [sic!] to «A sum which has the same value as its reciprocal»; Researcher: «Is it or isn't it a "circumference"?»; student A: «Absolutely not. A circumference must have x^2+y^2»; student B: «If it is simplified, yes» [i.e it is the semiotic transformation of treatment which gives or not a certain sense: the inverse operations would lead back to a … "circumference"];[3]

$$(n-1)+n+(n+1) \xrightarrow{\text{TREATMENT}} 3n$$

sense: from «The sum of three consecutive whole numbers» to «The triple of a natural number»; Researcher: «Is it possible to consider it the sum of three consecutive whole numbers?»; student C: «No*, like that*, no, *like that* it's the sum of three equal numbers, n".

The sum of the first 100 natural positive numbers (according to Gauss) is considered. The final semiotic result of successive changes effected via some treatments and conversions 101×50; this representation is not recognised as being a representation of the initial object; the presence of the multiplication sign forces all the students to search for a certain sense in mathematical objects in which the term "multiplication" (or a similar term) appears.

5.6. Graduated students

Trainee secondary school teachers

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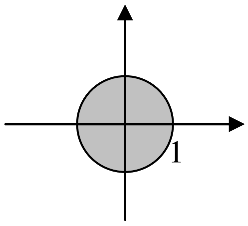
[3] But, as the reader should have already noted, here it is by no means a circumference.

Mathematical object: The sum of two square numbers is less than 1; semiotic representation universally shared: $x^2+y^2<1$; after changes in semiotic representation via treatments: $(x+iy)(x-iy)<1$ and conversion:

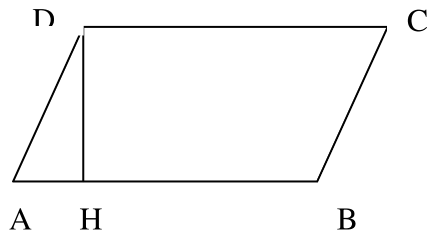


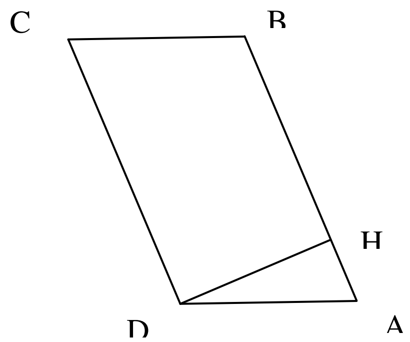
arriving at: $\rho^2+i^2<0$. In spite of that fact that the transformations are clearly and explicitly carried out, discussing each change of semiotic register, nobody is willing to admit the unique nature of the mathematical object in question. The final representation is considered a “parametric inequality in C”; the *sense* has been modified.

Trainee secondary school teachers
A) Mathematical object: Sequence of triangular numbers; interpretation and conversion: 1, 3, 6, 10, …; change of representation via treatment: 1, 1+2, 1+2+3, 1+2+3+4,….; this representation is seen as «Sequence of the partial sums of the succeeding natural numbers».
B) Mathematical object: Sequence of square numbers; interpretation and conversion: 0, 1, 4, 9, …; change of representation via treatment: 0, (0)+1, (0+1)+3, (0+1+3)+5,….; this representation is seen only as «Sum of the partial sums of the succeeding odd numbers».

In none of these quickly described examples did the students accept that the *sense* of the final semiotic representation obtained via the semiotic transformations illustrated coincided with the *sense* of the initial mathematical object. Such a result clearly indicates a path for future analysis.

5.7. Primary school teachers

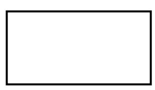


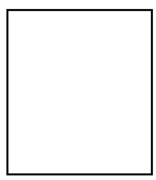


«DH is the height» «DH isn't the height»

5.8. Middle school teachers

From the text of a problem: «The height of a rectangle is 2/3 of the base, knowing…»;





«This figure represents the situation…» «… but this doesn't»; why? «Because here the base is shorter».

5.9. Secondary school teachers

«I can make a *bijective mapping of* N with Z, but Z has more elements than N».

6. Discussion of the representations of a given object provided by Primary school teachers and considered suitable for their pupils

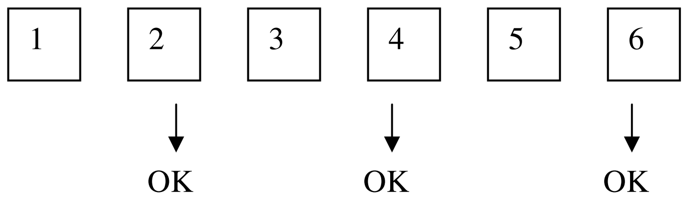
During a Primary school teacher training course we addressed the theme: *First elements of probability*. In conclusion we asked the teachers to represent the mathematical object: “the result of throwing a die is an even number”, using the symbolism they considered most suitable for their pupils. All the proposals were collected and voted on. The following results are in order of preference:

$\frac{3}{6}$ 50% $\frac{1}{2}$ (three and three) $\frac{\bullet\bullet\bullet}{\circ\circ\circ}$

(three out of six) $\frac{\circ\circ\circ}{6}$ (three out of six) $\frac{\circ\circ\circ}{\bullet\bullet\bullet\bullet\bullet\bullet}$

(2, 4, 6 in relation to 1, 2, 3, 4, 5, 6) $2\cdot 4\cdot 6 \big/ 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6$

(figurative - operative)



The importance of analysing the production of students is underlined by Duval: «Emphasis on the importance of descriptions in the acquisition of scientific knowledge and the first stages of mathematical learning must be accompanied by consideration of another question, fundamental both for research and for teachers: the analysis of the productions of pupils. It is within the perspective of the development of descriptions that the most personal and diverse productions are obtained, since these can be realised verbally or through drawings or diagrams… In this case, for research a methodological question is posed, while for teachers the question is one of diagnosis. We shall see how any analysis of students’ productions must clearly distinguish, for every semiotic production – discursive or non-discursive -, between *different levels of articulation of sense* which do not demonstrate the same operations» (Duval 2003).

In the considered case in this section 6, the "pupils" are Primary school teachers, while the "teachers" are university lecturers.[4]
The productions of the "pupils-teachers" previously illustrated at the beginning of this section can be analysed in a number of different ways. Once again I will follow the approach of Duval (2003): «(…) we must not confuse what we shall call an 'real' task of description with a 'merely formal' one. (…) A task of description is real when it requires observation of the object of the situation to describe. (…) In this case the pupil has access to each of the two elements of the pair (object and representation of the object) independently. A task of description is merely formal when it requires a simple change of register of representation: a verbal description based on a drawing or an "image" or vice versa. The pupil no longer has independent access to the object represented. Formal descriptions are thus conversion tasks designed to maintain invariable what is represented (…)» (Duval 2003).
I believe that this distinction proposed by Duval can explain, at least in part, the episode described in paragraphs 2 and 5.
When a mathematical object is observable and known through shared practices, the "real description" completely corresponds to the characteristics of the object, i.e. to the practice constructed around it and with it, and thus to the *sense* that all this acquires for participants in the elaboration of this practice. But the use of semiotic transformations at times leads to substantial modifications in the description, thereby "becoming a merely formal description" obtained via semiotic practices which may be shared, but which deny access to the object represented and so compromise the conservation of its *sense*.

7. Conclusion

What I would like to emphasize here is how the sense of a mathematical object is more complex than it is considered within the usual pair (object and its representations). There are semiotic links between pairs of this kind:

(object, its representation) – (object, its other representation)

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[4] That this "change of role" can be considered as plausible is amply demonstrated in the international literature; here I will indicate only what is furnished within the field of PME by Llinares, Krainer (2006), which contains a rich specific bibliografy.

These links are due to semiotic transformations between the representations of the same object, but then cause the loss of sense of the initial object. Although both object and semiotic transformations are the result of shared practices, the outcomes of the transformations can require *other* attributions of sense through *other* shared practices. This is highly suggestive for all studies of ontology and knowledge.

The phenomenon described in the second part of the article can be used to complete the picture proposed by Duval of the role of the multiple representations of an object in understanding it and also to break the vicious circle of his paradox. As a matter of fact every representation carries with it a *different* "subsystem of practices", from which emerge *different* objects (previously called O_1, O_2, O_3 y O_4). But the articulation of these objects within a more general system requires a change of perspective, a movement into another context in which the search for a *common structure* is a part of the system of global practices in which distinct "partial objects" play a role.

The progressive development of the use of different representations undoubtedly enriches the meaning, the knowledge and the understanding of the object, but also its complexity. In one sense the mathematical object presents itself as unique, in another as multiple.

What is then the nature of the mathematical object? The only reply would seem to be "structural, formal, grammatical" (in the epistemological sense) together with "global, mental, structural" (in the psychological sense) which we as subjects construct within our brains as our experience is progressively enriched.

Clearly these considerations lead to potential future developments in which ideas, apparently diverse, will work together to search for explanations for phenomena concerning the attribution of sense.

References

Blumer H. (1969). *Symbolic interactionism. Perspective and method.* Englewood Cliffs NJ: Prentice Hall.

Catena P. (1992). *Universa loca in logicam Aristetolis in mathematicas disciplinas*. (G. Dell'Anna ed.). Galatina (Le): Congedo.

Chevallard Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*. 12, 1, 73-112.

D'Amore B. (1999). *Elementi di didattica della matematica*. Preface by Colette Laborde. Bologna: Pitagora. [Version in Spanish: D'Amore B. (2006). *Didáctica de la Matemática*. With a letter by Guy Brousseau. Preface to the Spanish edition by Luis Rico. Bogotá: Editorial Magisterio].

D'Amore B. (2001a). Concettualizzazione, registri di rappresentazioni semiotiche e noetica. *La matematica e la sua didattica*. 2, 150-173. [Version in Spanish: D'Amore B. (2004). Conceptualización, registros de representaciones semióticas y noética: interacciones constructivisticas en el aprendizaje de los conceptos matemáticos e hipótesis sobre algunos factores que inhiben la devolución. *Uno*. 35, 90-106].

D'Amore B. (2001b). Un contributo al dibattito su concetti e oggetti matematici: la posizione "ingenua" in una teoria "realista" vs il modello "antropologico" in una teoria "pragmatica". *La matematica e la sua didattica*. 1, 4-30. [Version in Spanish: D'Amore B. (2001). Una contribución al debate sobre conceptos y objetos matemáticos. *Uno*. 27, 51-76].

D'Amore B. (2003a). La complexité de la noétique en mathématiques ou les raisons de la dévolution manquée. *For the learning of mathematics*. 23, 1, 47-51. [Version in Spanish: D'Amore B. (2002). La complejidad de la noética en matemáticas como causa de la falta de devolución. *TED*. Bogotà, Università Pedagogica Nazionale. 11, 63-71].

D'Amore B. (2003b). *Le basi filosofiche, pedagogiche, epistemologiche e concettuali della Didattica della Matematica*. Bologna: Pitagora. [Version in Spanish: D'Amore B. (2005). *Bases filosóficas, pedagógicas, epistemológicas y conceptuales de la Didáctica de la Matematica*. México DF, México: Reverté-Relme. Preface by Guy Brousseau. Preface to the Spanish version by Ricardo Cantoral. Translation by Martha Isabel Fandiño Pinilla]. [Version in Portugese: D'Amore B. (2005). *As bases filosóficas, pedagógicas, epistemológicas e conceituais da didáctica da matematica*. Preface to the Italian edition: Guy Brousseau. Preface: Ubiratan D'Ambrosio Tradução: Maria Cristina Bonomi Barufi. Escrituras: São Paulo].

D'Amore B., Godino D. J. (2006). Punti di vista antropologico ed ontosemiotico in Didattica della Matematica. *La matematica e la sua didattica*. 1, 7-36. [Version in Spanish: in preparation].

D'Amore B., Radford L., Bagni G.T. (2006). Ostacoli epistemologici e prospettiva socio-culturale. *L'insegnamento della matematica e delle scienze integrate*. 29B, 1, 11-40.
Daval R. (1951). *La métaphysique de Kant*. Paris: PUF.
Duval R. (1993). Registres de représentations sémiotiques et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*. 5, 37-65.
Duval R. (2003). Décrire, visualiser ou raisonner: quels 'apprentissages premiers' de l'activité mathématique? *Annales de Didactique et de Sciences Cognitives*. 8, 13-62.
Duval R. (2005). Transformations de representations semiotiques et demarche de pensée en mathématiques. Colloque COPIRELEM, Strasbourg, 30th may 2006 – 1st june 2006. In press.
Eco U. (1973). *Segno*. Milano: ISEDI.
Eco U. (1975). *Trattato di semiotica generale*. Milano: Bompiani.
Godino J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática. *Recherches en Didactique des Mathématiques*. 22, 2/3, 237-284.
Godino J. D., Batanero C. (1994). Significado istitucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques*. 14, 3, 325-355.
Godino J. D., Batanero C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In: Sierpinska A., Kilpatrick J. (eds.) (1988). *Mathematics Education as a Research Domain: A Search for Identity*. (177-195). Dordrecht: Kluwer A. P.
Kozoulin A. (1990). *Vygotsky's psychology*. Cambridge, MA: Harvard Univ. Press.
Llinares S., Krainer K. (2006). Mathematics (student) teachers and teacher educators as learners. In: Gutierrez A., Boero P. (eds.) (2006). *Handbook of Research on the Psychology of Mathematics Education.* Rotterdam, The Netherlands: Sense Publishers B.V. In press.
Radford L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational studies in mathematics*. 42, 3, 237-268.
Radford L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the learning of mathematics*. 22, 2, 14-23.
Radford L. (2003). Gestures, speech and the sprouting of signs. *Mathematical thinking and learning*. 5, 1, 37-70.

Radford L. (2004). Cose sensibili, essenze, oggetti matematici ed altre ambiguità. *La matematica e la sua didattica*. 1, 4-23.
Radford L. (2005). La generalizzazione matematica come processo semiotico. *La matematica e la sua didattica*. 2, 191-213.
Wertsch JV. (1991). *Voices in the mind. A sociocultural approach to mediate action*. Cambridge, MA: Harvard Univ. Press.
Zinchemko VP. (1985). Vygotsky's ideas about units for he analysis of mind. In: Wertsch J. V. (ed.) (1985). *Culture, communication and cognition: Vygotskian perspectives*. Cambridge, MA: Harvard Univ. Press.